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Chromatic Number of in-Regular Types of Halin Graphs

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Abstract: **A Halin graph H is the union of a tree $T \neq K_2$ with no vertex of degree two and a cycle C connecting the end-vertices of T in the cyclic order determined by a plane embedding of T. In this paper, we classify the Halin graphs depending upon whether the tree T is unicentric or bicentric and investigate the vertex coloring properties of four classes of Halin graphs.**

Keywords**: In-regular circular Halin graph, in-regular belted circular Halin graph, in-regular elliptical Halin graph, in-regular belted elliptical Halin graph.**

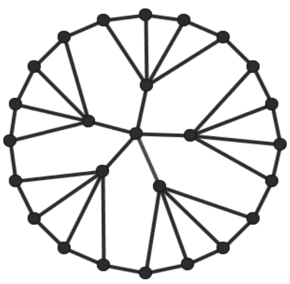
I. INTRODUCTION

A Halin graph is a plane graph $G = T \cup C$ where $T \neq K_2$ is a tree with no vertex of degree 2 and C is a cycle connecting the leaves of T in the cyclic order determined by the plane embedding of T. Halin Graphs belong to the family of planar 3-connected graphs and possess Hamiltonian properties. They are 1-Hamiltonian, (ie., they are Hamiltonian) and remain so even after the removal of any single vertex as given in Bondy [1]. In the recent years, many researchers have been studying the coloring [3, 9, 10] and list coloring [4] of Halin graphs. Recently some scholars begin to consider the adjacent vertex acyclic edge coloring of graphs [2], the adjacent vertex distinguishing edge coloring of planar graphs [5] and Halin graphs [7, 6]. We use the notation V(G) and E(G) for the vertex and the edge sets of G respectively.

In this paper we obtain the chromatic number of different types of Halin graphs depending upon whether it is unicentric or bicentric

A Halin graph $G = T \cup C$ in which the tree T has one vertex as its center, the number of levels ℓ, the degree of inner vertices D and outer vertices obviously having degree three is called the in-regular circular Halin graph and denoted by $H_1(\ell, D)$

Example: $H_1(2,5)$



The vertex set of $H_1(\ell, D)$ can be divided into two disjoint sets called inner nodes and outer nodes. The outer nodes are precisely the leaf nodes and inner nodes are the non-leaf nodes of T. It is noted that only the outer nodes are in the cycle C. The Halin graph in which the tree T has a star structure (ie. only one non-leaf node) is called a wheel.

Let $H_1(\ell, D)$ be a Halin graph and w be an inner node which is adjacent to only one other inner node. Define C(w) as the set of all outer nodes adjacent to the inner node w. The sub graph of H induced by w ∪ C(w) is referred as fan and w is called center of this fan.

For an in-regular circular Halin graph $H_1(\ell, D)$, some of the interesting aspects are:

1) The total number of vertices in $H_1(\ell,D)$ is

$$\begin{cases} 1+ D+ D(D-1)+ D(D-1)^2+ D(D-1)^3 + \ldots + D(D-1)^{\ell-1} & \text{for } \ell \geq 1 \\ 1, & \text{for } \ell = 0 \end{cases}$$

2) Every $H_1(\ell, D)$ is Hamiltonian.

3) The total number of leaves in $H_1(\ell, D)$ is $D(D-1)^{\ell-1}$.

4) The total number of fans in $H_1(\ell, D)$ is $D(D-1)^{\ell-2}$.

A proper vertex coloring of a graph G is an assignment of colors to the vertices of G one color to each vertex so that adjacent vertices are colored differently. The minimum number of colors required for the proper vertex coloring of the graph G is called chromatic number, denoted as $\chi(G)$.

II. MAIN RESULTS

A. Vertex colorings in in-regular circular Halin Graph:

Theorem 2.1

In $H_1(\ell, D)$, $D > 2$ with level $\ell = 1$, the chromatic number $\chi(H_1(\ell, D)) = \begin{cases} 4 \text{ if } D \text{ is odd} \\ 3 \text{ if } D \text{ is even} \end{cases}$

Proof.

In this case, $H_1(\ell, D)$ = a wheel. The result follows.

Theorem 2.2

In $H_1(\ell, D)$ where $D > 2$ with level $\ell \geq 2$, the chromatic number $\chi(H_1(\ell, D)) = 3$.

Proof.

$H_1(\ell, D)$ is an in-regular circular Halin graph with level $\ell \geq 2$ and degree $D > 2$ having one center u, say. Let the level ℓ of $H_1(\ell, D)$ be n and degree be D = m. Since it is unicentric, fix the color c_1 on the central vertex u where $\ell = 0$. The m vertices at level $\ell = 1$ are independent and are adjacent to u. Hence these m vertices receive a color c_2. If $\ell = k < n$, then the $m(m-1)^{k-1}$ vertices on the level $\ell = k$ are independent which can all be coloured by c_1 or c_2, whichever color is held by the vertices in the level k-2. Now, for any vertex w at the $(n-1)^{th}$ level, the fan induced by the vertices w U C(w) requires three colors, that is a color c_3 in addition to the two colors c_1 and c_2 already used. In this process, the vertices in the cycle C are alternatively colored with two colors other than that color assigned to w. Since the number of vertices $m(m-1)^{n-1}$ is even for any m, these two colors will exhaust all the leaf nodes on the cycle. This gives $\chi(H_1(\ell,D)) \leq 3$. Since $H_1(\ell, D)$ has a triangle as an induced sub graph, $\chi(H_1(\ell, D)) \geq 3$. Hence the result follows.

A Halin graph $H_1(\ell, D)$ in which the vertices of each level $0 < \ell < n$ are connected by a cycle, contributing degree 2 to each inner vertex such that the resulting graph maintains the inner degree D is called in-regular belted circular Halin graph and denoted by $BH_1(\ell, D)$.

Example: $BH_1(2,5)$

The in-regular belted circular Halin graph $BH_1(\ell, D)$ obviously holds the following interesting properties.

1) The total number of vertices in $BH_1(\ell, D)$, $D > 3$ is

$$\begin{cases} 1+D+D(D-3)+D(D-3)^2+D(D-3)^3+\ldots\ldots+D(D-3)^{\ell-1} & \text{for } \ell \geq 1 \\ 1 & \text{for } \ell = 0 \end{cases}$$

2) The total number of leaves in $BH_1(\ell, D)$, $D > 3$ is $D(D-3)^{\ell-1}$.

3) The total number of fans in $BH_1(\ell, D)]$, $D > 3$ is $D(D-3)^{\ell-2}$.

4) $BH_1(1, D)$ is same as $H_1(1, D)$, where $D > 2$.

Theorem 2.3

$BH_1(\ell, 3)$, where $\ell \geq 2$ does not exist.

Proof.

It is obvious that $BH_1(1, 3) = H_1(1, 3) =$ Wheel W_3. If $\ell \geq 2$, then in the in-regular belted circular Halin graph with uni-center, every vertex in any level k, where $0 < k < \ell$, has degree 4 which is a contradiction since $D = 3$.

Remark:

The graph $BH_1(\ell, D)$, where D is of odd and $D > 2$ with level $\ell = 1$ is a wheel on odd cycle for which the chromatic number $\chi(BH_1(\ell,D))=4$.In the case when D is even, $\chi(BH_1(\ell, D)) = 3$.

Theorem 2.4

For the graph $BH_1(\ell, D))$ where D is of even degree, $D > 2$ with level $\ell>1$, the chromatic number $\chi(BH_1(\ell, D)) = 3$.

Proof.

Obviously $\chi(BH_1(\ell, D)) \geq 3$ since it has a triangle as an induced sub graph. Fix a color c_1 for the central vertex u. The vertices at the first level can be alternatively colored with c_2 and c_3. Each inner vertex v emanates exactly D-3 vertices which are belted (that is, which lie in a cycle on $D(D-3)^k$ vertices for some k), where D-3 is odd. The vertex v, together with the (D-3) vertices adjacent to v, induces a fan. To achieve a 3-coloring, the (D-3) vertices in any such fan in the inner cycle must have colors in one of the following schemes:

a) $c_1, c_2, c_1, c_2, \ldots, c_1, c_2, c_1$.

b) $c_2, c_3, c_2, c_3, \ldots, c_2, c_3, c_2$.

c) $c_3, c_1, c_3, c_1, \ldots, c_3, c_1, c_3$.

Let $w_1, w_2, \ldots, w_{D-3}$ be the (D-3) vertices of the fan emanating from (adjacent to) v.

Case a.1. v is a vertex in Scheme (a).

If v has the color c_1, then $w_1, w_2, \ldots, w_{D-3}$ are colored by Scheme (b).

If v has the color c_2, then $w_1, w_2, \ldots, w_{D-3}$ are colored by Scheme (c).

Case b.1. v is a vertex in Scheme (b).

If v has the color c_2, then $w_1, w_2, \ldots, w_{D-3}$ are colored by Scheme (c).

If v has the color c_3, then $w_1, w_2, \ldots, w_{D-3}$ are colored by Scheme (a).

Case c.1. v is a vertex in Scheme (c).

If v has the color c_3, then $w_1, w_2, \ldots, w_{D-3}$ are colored by Scheme (a).

If v has the color c_1, then $w_1, w_2, \ldots, w_{D-3}$ are colored by Scheme (b).

This color scheme can be extended to all the levels of the graph and a 3-coloring can be achieved.

Theorem 2.5

In $BH_1(\ell,D)$ where D is of odd degree, $D \geq 5$ with level $\ell>1$, chromatic number $\chi(BH_1(\ell, D)) = 4$.

Proof:

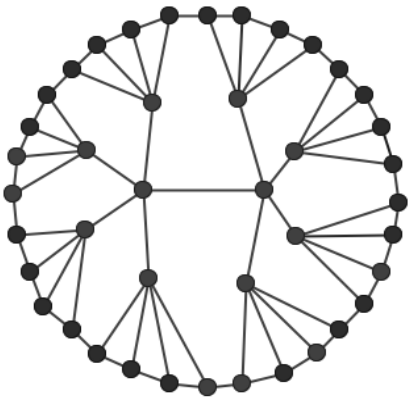
Obviously $\chi(BH_1(\ell, D)) \geq 4$ since the union of the central vertex and the D vertices at level $\ell = 1$ induce a wheel on odd cycle. Let c_1, c_2, c_3 and c_4 be any four colors. Fix c_1 for the central vertex u. The vertices at the first level can be alternatively colored with c_2 and c_3 and the D^{th} vertex with c_4, since the inner cycle at level $\ell = 1$ is of odd length. Each inner vertex v emanates exactly D-3 vertices which are belted (that is, which lie in a cycle on $D(D-3)^k$ for some k), where D-3 is even. The vertex v, together with the (D-3) vertices adjacent to v, induces a fan. To achieve a 4-coloring, the (D-3) vertices in any such fan in the inner cycle can be colored by the scheme as in Theorem 2.4, but with three colors.

Hence 4-coloring can be achieved

B. Vertex colorings in in-regular Elliptical Halin Graph:

A Halin graph in which the tree has two vertices as its centers, ℓ the number of levels, D the degree of inner vertices and the outer vertices having degree three is called an in-regular elliptical Halin graph and denoted by $H_2(\ell, D)$.

Example:$H_2$2,5)



The in-regular elliptical Halin graph $H_2(\ell, D)$ has the following properties.

1) The total number of vertices in $H_2(\ell,D)$ is

$$\begin{cases} 2[1+(D-1)+(D-1)^2+(D-1)^3+\ldots+(D-1)^{\ell}] & \text{for } \ell \geq 1 \\ 2 & \text{for } \ell=0 \end{cases}$$

2) Every $H_2(\ell, D)$ is Hamiltonian.

3) The total number of leaves in $H_2(\ell, D)$ is $2(D-1)^{\ell}$.

4) The total number of fans in $H_2(\ell, D)$ is $2(D-1)^{\ell-1}$.

Theorem 3.1

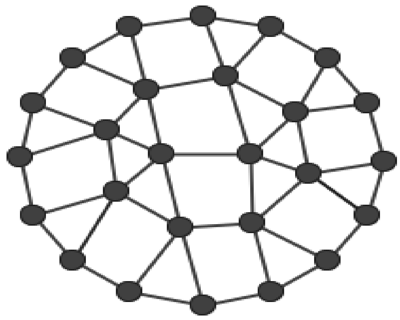
In $H_2(\ell, D)$ where D >2 with level $\ell \geq 1$, the chromatic number $\chi(H_2(\ell, D)) = 3$.

Proof.

Let $H_2(\ell, D)]$ be an in-regular elliptical Halin graph with $\ell \geq 1$, D>2. Fix the colors c_1 and c_2 to the two centers u and v respectively of the underlying tree of the graph. Let A be the sub tree rooted at u which has level ℓ. Consider the sub tree $T_1 = A - uv$. T_1, being a tree, is 2-colorable, say with colors c_1 and c_2 up to the level *ℓ-1*. Similarly, $T_2 = B - uv$ is also 2-colorable, say with the same colors c_1 and c_2 up to the level *ℓ-1*, where B is the sub tree rooted at v having level ℓ. Then $T = T_1 \cup T_2 \cup uv$. All the leaf nodes of T lie on the cycle C. Since every inner vertex at level *ℓ-1*, together with the V(C), induces a fan which requires an additional color, say c_3, in addition to c_1 and c_2. Hence $\chi(H_2(\ell, D)) \leq 3$. Since $H_2(\ell, D)$ has an odd cycle as an induced sub graph, $\chi(H_2(\ell, D)) \geq 3$. Hence the result follows.

A Halin graph $H_2(\ell, D)$ in which the vertices of each level $0 < \ell < n$ are connected by a cycle, contributing degree 2 to each inner vertex such that the resulting graph maintains the inner degree D is called in-regular belted elliptical Halin graph and denoted by $BH_2(\ell, D)$.

Example: $BH_2(2,5)$



For an in-regular belted elliptical Halin graph, $BH_2(\ell, D)$, some of the interesting aspects are:

1) For $\ell > 2$ and $D = 3$, $BH_2(\ell, D)$ does not exist.

2) The total number of vertices in $BH_2(\ell, D)$, $D > 3$ is

$$\begin{cases} 2[1+(D-1)+(D-1)(D-3)+(D-1)(D-3)^2+\dots+(D-1)(D-3)^{\ell-1}] & \text{for } \ell \geq 1 \\ 2 & \text{for } \ell = 0 \end{cases}$$

3) Every $BH_2(\ell, D)$ is Hamiltonian.

4) The total number of leaves in $BH_2(\ell, D)$ is $2(D-1)(D-3)^{\ell-1}$.

5) The total number of fans in $BH_2(\ell, D)$ is $2(D-1)(D-3)^{\ell-2}$.

Theorem 3.2

$BH_2(\ell, D)$, where $D = 3$ with level $\ell \geq 2$, does not exist.

Proof:

Let u and v be the centers of $BH_2(\ell, D)$. Let $x \neq u$ or v be any inner vertex. Since the degree of any inner vertex other than the center is at least 4 in any belted graph $BH_2(\ell, D)$, it concludes that $BH_2(\ell, D)$ does not exist.

Theorem 3.3

In $BH_2(\ell, D)$, where $D > 3$ with level $\ell = 1$, the chromatic number $\chi(BH_2(\ell, D)) = 3$.

Proof:

Let u and v be the centers of $BH_2(\ell, D)$. Assign the colors c_1 and c_2 to vertices u and v respectively. Since $\ell = 1$, the center u is adjacent to D-1 outer vertices on the outer cycle which can be alternately colored with c_2 and c_3 in order. Similarly, the other centre v is also adjacent to D-1 outer vertices on the outer cycle which are colored alternately with c_1 and c_3. Since there are only 2D-2 vertices on the outer cycle, the 3-coloring on the cycle is proper, which proves the result.

Theorem 3.4

In $BH_2(\ell, D)$ where $D > 3$ with level $\ell > 1$, the chromatic number $\chi(BH_2(\ell, D)) = 3$.

Proof:

Extending the 3-coloring of the vertices at level $\ell = 1$ obtained in Theorem 3.3, to the subsequent levels, a 3-coloring is achieved for the vertices of the graph $BH_2(\ell, D)$. Hence $\chi(BH_2(\ell, D)) = 3$.

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